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Boyd, Thornton

Monterey, California ; Naval Postgraduate School

A METHOD OF APPROXIMATING PERCENTILE POINTS OF HIGHLY SKEWED, TWO PARAMETER BETA DISTRIBUTIONS

by

Thornton Boyd

United States
Naval Postgraduate School

THESIS

A METHOD OF APPROXIMATING PERCENTILE
POINTS OF HIGHLY SKEWED, TWO
PARAMETER BETA DISTRIBUTIONS

by

Thornton Boyd

Thesis Advisor: W. M. Woods

March 1971

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by

Thornton Boyd
Captain, United States Marine Corps
B.A., University of Texas, 1963

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ABSTRACT

An approximation formula is derived which provides a simplified and efficient method of calculating the αth percentile point of a two parameter beta function when the two parameters are known. A specific application is presented by utilizing the formula to compute the lower $100(1-\alpha)\%$ Bayesian confidence limit for the reliability of a component when the prior distribution of that reliability is known to be beta with parameters a' and b'. The posterior distribution is then determined by mission testing n items and recording the number of successes, s. This distribution is known to be beta with parameters a and b, where $a = a' + s$ and $b = b' + n - s$. Therefore, the formula can be utilized to determine the αth percentile point of this posterior distribution which by definition is the lower $100(1-\alpha)\%$ confidence limit for the reliability of the component.

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I. INTRODUCTION

The object of this paper was to determine a method of approximating the αth percentile point for the reliability of a component when this reliability is known to have a beta distribution. More specifically, the prime interest was in approximating percentile points of highly skewed, two parameter beta distributions. The approximation formula that was derived provides a simplified method for determining the αth percentile point of a beta distribution when the two parameters a and b are known.

Given that the reliability R of a component is distributed beta with parameters a' and b' the posterior distribution of R can then be determined.

Let n = the number to be sampled

Let s = the number of successes

Let f = the number of failures

The n items are then mission tested. By applying Bayes Theorem, the posterior distribution of R is found to be beta with parameters (a' + s) and (b' + f).

Let a = a' + s and b = b' + f

Therefore, the posterior distribution of R is beta with parameters a and b. Once the values of a and b have been determined, the formula below can be utilized to approximate p_α, the αth percentile point of the reliability of the component.

$$p_\alpha \doteq \left[\text{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \text{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{3(a+b-1)}$$

Where $Z_{1-\alpha}$ is a number such that $P\left[Z > Z_{1-\alpha}\right] = 1 - \alpha$ and Z is distributed normal (0,1). Therefore, $Z_{1-\alpha}$ can be looked up in any Standard Normal Tables.

II. DISCUSSION

A. BAYESIAN RELIABILITY

Given that the prior density, $h_R(r;a',b')$, is known and that the conditional density, $g_{S|R}(s|r)$, has been determined by testing, the posterior density, $f_{R|S}(r|s)$, can be computed by use of Bayes Theorem:

$$f_{R|S}(r|s) = \frac{g_{S|R}(s|r)\ h_R(r;a',b')}{\int_{-\infty}^{\infty} g_{S|R}(s|r)\ h_R(r;a',b')dr} .$$

Once the posterior density has been determined the $100(1-\alpha)\%$ Lower Bayesian Confidence Limit, $\hat{R}_{L(\alpha)}$, is defined by,

$$P\left[\hat{R}_{L(\alpha)} \leq R\right] = 1-\alpha$$

where R is the reliability of the component. Therefore, $\hat{R}_{L(\alpha)}$ is the αth percentile point of the posterior distribution of R.

B. APPROXIMATING A BINOMIAL DISTRIBUTION WITH A BETA DISTRIBUTION

The following identity relates the binomial distribution with the beta distribution. If U is distributed beta with parameters s and $(n - s + 1)$, where n and s are positive integers and $s \leq n$,

$$P(U \leq p) = \sum_{j=s}^{n} \binom{n}{j} p^j (1-p)^{n-j}, \quad \text{for } 0 \leq p \leq 1.$$

Therefore, $P(U \leq p) = P(X \geq s)$ where X is distributed binomial with parameters n and p.

C. THE POSTERIOR DISTRIBUTION

Given that the prior density, $h_R(r;a',b')$ is beta with parameters a' and b' then

$$h_r(r;a',b') = \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\, r^{a'-1}(1-r)^{b'-1}, \text{ for } 0 \leq r \leq 1.$$

The conditional density, $g_{S|R}(s|r)$, is binomial with parameters n and r where

n = the number of items to be mission tested

r = the probability of success

s = the number of successes observed.

Therefore, the conditional density is

$$g_{S|R}(s|r) = \binom{n}{s} r^s (1-r)^{n-s}.$$

The posterior density, $f_{R|S}(r|s)$ is given by

$$f_{R|S}(r|s) = \frac{g_{S|R}(s|r)\; h_R(r;a',b')}{\int_0^1 g_{S|R}(s|r)\; h_R(r;a',b')\,dr}$$

$$= \frac{\binom{n}{s} r^s (1-r)^{n-s} \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\, r^{a'-1}(1-r)^{b'-1}}{\int_0^1 \binom{n}{s} r^s (1-r)^{n-s} \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\, r^{a'-1}(1-r)^{b'-1}\, dr}$$

$$= \frac{r^{a'+s-1}(1-r)^{b'+n-s-1}}{\int_0^1 r^{a'+s-1}(1-r)^{b'+n-s-1}\, dr}$$

$$= \frac{r^{a'+s-1}\;(1-r)^{b'+n-s-1}}{\frac{\Gamma(a'+s)\Gamma(b'+n-s)}{\Gamma(a'+b'+n)}}$$

$$= \frac{\Gamma(a'+b'+n)}{\Gamma(a'+s)\Gamma(b'+n-s)} r^{a'+s-1}(1-r)^{b'+n-s-1} .$$

Therefore, the posterior density is beta with parameters $(a' + s)$ and $(b' + n - s)$. Let $a = a' + s$ and $b = b' + n - s$. Therefore, the posterior distribution of R is beta with parameters a and b.

III. THE APPROXIMATION FORMULA

A. DERIVATION

If X is binomial with parameters n and p, then $Y = 2 \text{ Arcsin} \sqrt{\frac{X}{n}}$ in radians is approximately normally distributed with parameters μ and σ^2, where $\mu = 2 \text{ Arcsin} \sqrt{p}$ and $\sigma^2 = \frac{1}{n}$.[1]

$$E\left[\frac{X}{n}\right] = p$$

$$P\left[\frac{X}{n} \le y\right] \simeq \int_{-\infty}^{2 \text{ Arcsin} \sqrt{y + \frac{1}{2n}}} p_Y(x)\, dx$$

$$P_Y(y) \simeq \Phi\left(\frac{2 \text{ Arcsin} \sqrt{y + \frac{1}{2n}} - 2 \text{ Arcsin} \sqrt{p}}{\frac{1}{\sqrt{n}}}\right)$$

Where $\frac{1}{2n}$ is a correction for continuity analogous to that used with the normal approximation.

If U is distributed beta with parameters s and $(n - s + 1)$, then

$$P[U \le p] = \sum_{j=s}^{n} \binom{n}{j} p^j (1-p)^{n-j} = P[X \ge s]$$

and $$P\left[\frac{X}{n} \ge \frac{s}{n}\right] = P[U \le p]$$

where X is binomial with parameters n and p.

$$P[U \le p] = P\left[\frac{X}{n} \ge \frac{s}{n}\right]$$

$$= P\left[2 \text{ Arcsin} \sqrt{\frac{X}{n}} \ge 2 \text{ Arcsin} \sqrt{\frac{s}{n}}\right]$$

$$= P\left[Z \ge \left(2 \text{ Arcsin} \sqrt{\frac{s}{n}} - 2 \text{ Arcsin} \sqrt{p}\right) \sqrt{n}\right]$$

---

[1] Brownlee, K.A., Statistical Theory and Methodology in Science and Engineering, p. 115-116, Wiley, 1960

Where $p = E\left[\frac{X}{n}\right]$ and Z is distributed normal (0,1). Now use $\frac{1}{2n}$ as a continuity correction factor.

$$P[U \le p] \doteq P[Z \ge (2 \text{ Arcsin} \sqrt{\frac{s}{n} + \frac{1}{2n}} - 2 \text{ Arcsin} \sqrt{p}) \sqrt{n}\ ]$$

$$= P[Z \le (2 \text{ Arcsin} \sqrt{p} - 2 \text{ Arcsin} \sqrt{\frac{s}{n} + \frac{1}{2n}}\ ) \sqrt{n}\ ]$$

If U is beta with parameters a and b, where a = s and b = n - s + 1.

$$b = n - s + 1$$

$$b = n - a + 1$$

$$n = a + b - 1$$

$$\frac{s}{n} + \frac{1}{2n} = \frac{2s + 1}{2n} = \frac{2a + 1}{2(a + b - 1)} = \frac{a + \frac{1}{2}}{a + b - 1}$$

therefore,

$$P[U \le p] \doteq P\left[Z \le \left(2 \text{ Arcsin} \sqrt{p} - 2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\right) \sqrt{a+b-1}\right]$$

and

$$\sigma^2 = \frac{1}{n} = \frac{1}{a+b-1}\ .$$

Therefore, 2 Arcsin U is approximately normally distributed

$$N\left(2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\ ,\ \frac{1}{a+b-1}\right).$$

In Bayesian analysis we want p such that $P[U > p] = 1-\alpha$, $0 \le \alpha \le 1$.

Then p is the lower $100(1-\alpha)\%$ Bayesian Confidence Interval on U.

$$P[U > p] \doteq P[Z > (2 \text{ Arcsin} \sqrt{p} - 2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\ ) \sqrt{a+b-1}\ ]$$

Therefore,

$$(2 \text{ Arcsin} \sqrt{p} - 2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\ ) \sqrt{a+b-1} \doteq Z_{1-\alpha}$$

where $Z_{1-\alpha}$ is a number that $p[Z > Z_{1-\alpha}] = 1-\alpha$ and can be looked up in any Standard Normal Table.

$$\left(2 \text{ Arcsin} \sqrt{p} - 2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\right)\sqrt{a+b-1} = Z_{1-\alpha}$$

$$2 \text{ Arcsin} \sqrt{p} = \frac{Z_{1-\alpha}}{\sqrt{a+b-1}} + 2 \text{ Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}$$

$$\sqrt{p} = \text{Sin}\left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \text{Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\right)$$

$$\therefore p \doteq \left[\text{Sin}\left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \text{Arcsin} \sqrt{\frac{a+\frac{1}{2}}{a+b-1}}\right)\right]^2$$

For simplicity, the fraction ½ was dropped from the numerator of the second term. This had the effect of reducing the approximated value of p. To compensate for this reduction, a correction factor was added to the formula as shown below:

$$p \doteq \left[\text{Sin} \quad \frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \text{Arcsin} \sqrt{\frac{a}{a+b-1}}\right]^2 + \frac{1}{2(a+b-1)}$$

However, if $b < 1.0$ then $\frac{a}{a+b-1} > 1.0$ and the Arcsin is not defined when the argument is greater than one. To prevent this problem, so that the approximation formula can be used for all values of b, the second term was modified by adding 1.0 to the denominator. After comparing the approximated values with the computed ones, the correction factor was also modified by multiplying the denominator by 3 instead of 2. This resulted in reducing the approximation to just below the computed value of the αth percentile points for low values of b, instead of just above it, as it was found to be originally. This change also caused the approximated values to be more uniformly accurate throughout the range of b.

Therefore, the final approximation formula for the αth percentile points of the beta distribution is:

$$P_\alpha \doteq \left[\operatorname{Sin}\left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin}\sqrt{\frac{a}{a+b}}\right)\right]^2 + \frac{1}{3(a+b-1)}$$

B. ACCURACY RESULTS

The αth percentile points generated by the approximation formula compare very favorably with the computed values which will hereafter be referred to as the "exact" values. As shown in Tables I, II and III, the approximation formula results in extremely accurate values for the percentile points of the beta distribution, especially for values of b less than or equal to five. Examination of the results indicates that the approximated value is less than the exact value when b values are less than 2.5 and greater when b values are greater than 2.5. Therefore, for values of b near 2.5, the approximation formula results in values which are very nearly the same as the exact values. The accuracy of the approximation is also a function of the parameter a. As the value of a increases the approximation becomes even more accurate. After using the approximation formula to compute various percentile points over a wide range of a and b values, it was noted that the worst estimated value was less than three thousandths away from the exact value for values of a greater than 40. Therefore, the formula will produce values for the αth percentile points of two parameter beta distributions with an accuracy which should be completely sufficient for most applications.

TABLE I. Comparison of Approximate and Exact 5th Percentile Points for the Beta Distribution

a \ b	0.5			2.5			5.0		
	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF
10	.81622	.82131	-.00509	.60786	.59059	+.01727	.47399	.46000	+.01399
20	.90394	.90735	-.00341	.77101	.76559	+.00642	.66533	.65819	+.00714
30	.93503	.93748	-.00245	.83863	.83517	+.00346	.75541	.75069	+.00472
40	.95091	.95283	-.00192	.87547	.87295	+.00252	.80743	.80392	+.00351
50	.96056	.96220	-.00164	.89863	.89665	+.00198	.84124	.83850	+.00274
60	.96704	.96836	-.00132	.91453	.91290	+.00163	.86496	.86266	+.00230
70	.97169	.97284	-.00115	.92612	.92473	+.00139	.88253	.88056	+.00197
80	.97519	.97620	-.00101	.93495	.93374	+.00121	.89605	.89434	+.00171
90	.97792	.97883	-.00091	.94189	.94082	+.00107	.90678	.90526	+.00152
100	.98011	.98083	-.00072	.94749	.94654	+.00095	.91551	.91416	+.00135

TABLE II. Comparison of Approximate and Exact 10th Percentile Points for the Beta Distribution

b / a	0.5			2.5			5.0		
	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF
10	.86285	.87052	-.00767	.66016	.64954	+.01062	.52241	.50796	+.01445
20	.92864	.93381	-.00517	.80384	.80026	+.00358	.70019	.69412	+.00607
30	.95180	.95555	-.00375	.86233	.86048	+.00185	.78203	.77851	+.00352
40	.96361	.96654	-.00293	.89398	.89283	+.00115	.82886	.82648	+.00238
50	.97078	.97320	-.00242	.91381	.91300	+.00081	.85915	.85670	+.00245
60	.97558	.97761	-.00203	.92740	.92688	+.00052	.88034	.87897	+.00137
70	.97903	.98079	-.00176	.93728	.93680	+.00048	.89600	.89487	+.00113
80	.98163	.98319	-.00156	.94480	.94440	+.00040	.90803	.90709	+.00094
90	.98365	.98504	-.00139	.95071	.95038	+.00033	.91757	.91676	+.00081
100	.98527	.98653	-.00126	.95547	.95519	+.00028	.92532	.92461	+.00071

TABLE III. Comparison of Approximate and Exact 20th Percentile Points for the Beta Distribution

a \ b	0.5			2.5			5.0		
	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF	APPROX	EXACT	DIFF
10	.91324	.91927	-.00603	.72149	.71152	+.00997	.58108	.56628	+.01480
20	.95485	.95928	-.00443	.84124	.83876	+.00248	.74106	.73577	+.00529
30	.96950	.97278	-.00328	.88908	.88818	+.00090	.81287	.81021	+.00266
40	.97697	.97955	-.00258	.91478	.91442	+.00036	.85355	.85196	+.00159
50	.98151	.98362	-.00211	.93081	.93069	+.00012	.87971	.87867	+.00104
60	.98455	.98635	-.00180	.94177	.94176	+.00001	.89795	.89721	+.00074
70	.98673	.98825	-.00152	.94973	.94978	-.00005	.91139	.91084	+.00055
80	.98837	.98976	-.00139	.95578	.95587	-.00009	.92170	.92128	+.00042
90	.98965	.99089	-.00124	.96053	.96063	-.00010	.92986	.92953	+.00033
100	.99068	.99180	-.00112	.96436	.96417	-.00011	.93648	.93622	+.00026

This is especially true for values of the 20th percentile points, Table III, for which the approximated values are exceptionally accurate.

C. ALTERNATE APPROXIMATION FORMULAS

If the approximation formula is used with the continuity correction factor that was originally derived,

$$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{2(a+b-1)} .$$

the resulting percentile points will all be greater than the exact values. The accuracy of the approximated values is extremely good for $b \leq 1.0$, but as the value of b increases, the approximated values exceed the exact values by an ever increasing amount. As a result, the approximated values become unacceptable when compared with the values generated by the formula using $\frac{1}{3(a+b-1)}$ as the continuity correction factor. See Table IV for a comparison of the values generated by the two formulas.

When the continuity correction factor is decreased by using $4(a+b-1)$ for the denominator, the approximated values will be less than the exact values. This trait could be useful if conservative estimates were desired. The approximation formula

$$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{4(a+b-1)}$$

is most accurate for values of $b \geq 4.0$ due to the fact that, as the value of b increases, the approximated value of p increases

with respect to the exact value. However, the approximated percentile points for lower values of b are not nearly as accurate as those given by the chosen formula. See Table IV for a comparison of the values generated by the two formulas.

The approximation formula can be simplified by not subtracting 1.0 from the denominators of the first term and the correction factor. The resulting formula,

$$p = \left[\mathrm{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b}} + \mathrm{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{3(a+b)}$$

gives very accurate approximations of the percentile points and is quite acceptable for most applications. This formula is most accurate for $b \leq 2.5$ and should be used whenever b is restricted to these values. The chosen formula is only superior in that it results in extremely accurate estimates for all values of b, not just very low values. However, the formula presented above would actually be preferred whenever b is restricted to very low values. See Table IV for a comparison of the values generated by the two formulas.

TABLE IV. Comparison of Approximate and Exact 20th Percentile Points Generated by Selected Approximation Formulas for the Beta Distribution

a	b	Exact	F-1	F-2	F-3	F-4
50	.5	.98362	.98151	.98487	.97982	.98156
50	1.0	.96832	.96719	.97052	.96552	.96728
50	1.5	.95486	.95429	.95759	.95264	.95442
50	2.0	.94242	.94225	.94552	.94061	.94240
50	2.5	.93069	.93081	.93405	.92920	.93099
50	3.0	.91950	.91987	.92307	.91826	.92006
50	3.5	.90876	.90933	.91250	.90774	.90953
50	4.0	.89840	.89915	.90229	.89758	.89936
50	4.5	.88838	.88929	.89240	.88773	.88951
50	5.0	.87867	.87971	.88280	.87817	.87995
100	.5	.99180	.99068	.99236	.98984	.99069
100	1.0	.98403	.98337	.98504	.98254	.98339
100	1.5	.97711	.97670	.97836	.97588	.97674
100	2.0	.97064	.97040	.97205	.96958	.97045
100	2.5	.96447	.96436	.96600	.96354	.96440
100	3.0	.95851	.95850	.96014	.95769	.95855
100	3.5	.95273	.95281	.95443	.95200	.95286
100	4.0	.94711	.94725	.94887	.94644	.94731
100	4.5	.94160	.94181	.94342	.94101	.94188
100	5.0	.93622	.93648	.93808	.93568	.93655

F-1 $$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{3(a+b-1)}$$

F-2 $$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{2(a+b-1)}$$

F-3 $$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{4(a+b-1)}$$

F-4 $$p = \left[\operatorname{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b}} + \operatorname{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{3(a+b)}$$

IV. EXAMPLES

A. EXAMPLE 1

Find the 20th percentile point for a = 100.0 and b = 2.0. Assume a' = 51.0 and b' = 1.0. A random sample of 50 items was then mission tested and one failure was observed.

$$n = 50.0 \text{ and } s = 49.0$$

$$a = a' + s = 51.0 + 49.0 = 100.0$$

$$b = b' + n - s = 1.0 + 50.0 - 49.0 = 2.0$$

$$\alpha = .20 \text{ and } Z_{1-\alpha} = Z_{.80} = -.842$$

$$P_{\alpha} = \left[\operatorname{Sin}\left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \operatorname{Arcsin}\sqrt{\frac{a}{a+b}}\right)\right]^2 + \frac{1}{3(a+b-1)}$$

$$P_{.20} = \left[\operatorname{Sin}\left(\frac{-.842}{2\sqrt{100+2-1}} + \operatorname{Arcsin}\sqrt{\frac{100}{100+2}}\right)\right]^2 + \frac{1}{3(100+2-1)}$$

$$= [\operatorname{Sin}(-.04189 + 1.43031)]^2 + .00330$$

$$= (.98341)^2 + .00330 = .96709 + .00330 = .97039$$

The table value is .97064 and the difference is only -.00025.

B. EXAMPLE 2

Find the 25th percentile point for a = 50.0 and b = 2.5. Assume a' = 26.0 and b' = 1.5. A random sample of 25 items was then mission tested and one failure was observed.

$$n = 25.0 \text{ and } s = 24.0$$

$$a = a' + s = 26.0 + 24.0 = 50.0$$

$$b = b' + n - s = 1.5 + 25.0 - 24.0 = 2.5$$

$$\alpha = .25 \text{ and } Z_{1-\alpha} = Z_{.75} = -.675$$

$$P_\alpha = \left[\text{Sin} \left(\frac{Z_{1-\alpha}}{2\sqrt{a+b-1}} + \text{Arcsin} \sqrt{\frac{a}{a+b}} \right) \right]^2 + \frac{1}{3(a+b-1)}$$

$$P_{.25} = \left[\text{Sin} \left(\frac{-.675}{2\sqrt{50+2.5-1}} + \text{Arcsin} \sqrt{\frac{50}{50+2.5}} \right) \right]^2 + \frac{1}{3(50+2.5-1)}$$

$$= [\text{Sin}\,(-.04703 + 1.35088)]^2 + .00647$$

$$= (.96456)^2 + .00647 = .93038 + .00647 = .93685$$

The table value is .93680 and the difference is only +.00005.

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Thornton Boyd

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13. ABSTRACT

An approximation formula is derived which provides a simplified and efficient method of calculating the αth percentile point of a two parameter beta function when the two parameters are known. A specific application is presented by utilizing the formula to compute the lower $100(1-\alpha)\%$ Bayesian confidence limit for the reliability of a component when the prior distribution of that reliability is known to be beta with parameters a' and b'. The posterior distribution is then determined by mission testing n items and recording the number of successes, s. This distribution is known to be beta with parameters a and b, where $a = a' + s$ and $b = b' + n - s$. Therefore, the formula can be utilized to determine the αth percentile point of this posterior distribution which by definition is the lower $100(1-\alpha)\%$ confidence limit for the reliability of the component.

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c.1 A method of approximating percentile points of highly skewed, two parameter beta distributions.